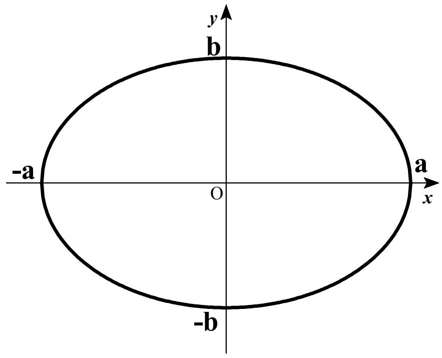
Area enclosed by an ellipse

1. Rectangular equation



The standard form : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The curve is symmetric about both the x and y axes.
We need to find the area in the first quadrant
and multiply the result by 4 .

$$\text{Area} = 4\int_0^a y\,dx = 4\int_0^a \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}dx = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

Put $x = a\sin\theta$. $dx = a\cos\theta\, d\theta$. When $x = a$, $\theta = \pi/2$. When $x = 0$, $\theta = 0$.

$$\therefore\quad \text{Area} = \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta d\theta) = 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}d\theta$$

$$= 2ab\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)d\theta = 2ab\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 2ab\left[\frac{\pi}{2}\right] = \underline{\underline{\pi ab}}$$

2. Parametric equation

(a) $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$, $0 \le t < 2\pi$.

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} \left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_0^{2\pi} [(a\cos t)(b\cos t)-(b\sin t)(-a\sin t)]dt$$

$$= \frac{1}{2}ab\int_0^{2\pi} [\cos^2 t+\sin^2 t]dt = \frac{1}{2}ab\int_0^{2\pi} dt = \frac{1}{2}ab(2\pi) = \underline{\underline{\pi ab}}$$

(b) $x = a\frac{1-t^2}{1+t^2}$, $y = b\frac{2t}{1+t^2}$, $-\infty < t < \infty$.

$$\text{Area} = \frac{1}{2}\int_{-\infty}^{\infty} \left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt = \frac{1}{2}\int_{-\infty}^{\infty} x^2\frac{d}{dt}\left(\frac{y}{x}\right)dt$$

$$= \frac{ab}{2}\int_{-\infty}^{\infty} \left(\frac{1-t^2}{1+t^2}\right)^2 \frac{2(1+t^2)}{(1-t^2)^2}dt = \frac{ab}{2}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \frac{ab}{2}\tan^{-1}t\Big|_{-\infty}^{\infty} = \underline{\underline{\pi ab}}$$

3. Polar equation

By putting $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get the polar form : $r^2 = \frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}$.

Area

$$= 4\times\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta = = 2\int_0^{\frac{\pi}{2}} \frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}d\theta = 2a^2b^2\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{a^2\tan^2\theta+b^2}d\theta = 2a^2b^2\int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{d(\tan\theta)}{a^2\tan^2\theta+b^2}$$

$$= 2a^2b^2\int_0^{\infty} \frac{du}{a^2u^2+b^2} = 2a^2b^2\left[\frac{1}{ab}\tan^{-1}\frac{au}{b}\right]_0^{\infty} = 2a^2b^2\left[\frac{1}{ab}\frac{\pi}{2}\right] = \underline{\underline{\pi ab}}$$